Spinorial geometry and Killing spinor equations of 6-D supergravity

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Abstract

We solve the Killing spinor equations of 6-dimensional (1,0)-supergravity coupled to any number of tensor, vector and scalar multiplets in all cases. The isotropy groups of Killing spinors are $Sp(1)\cdot Sp(1)\ltimes \mathbb{H}(1)$, $U(1)\cdot Sp(1)\ltimes \mathbb{H}(2)$, $Sp(1)\ltimes \mathbb{H}(3,4)$, $Sp(1)(2)$, $U(1)(4)$ and $\{1\}(8)$, where in parenthesis is the number of supersymmetries preserved in each case. If the isotropy group is non-compact, the spacetime admits a parallel null 1-form with respect to a connection with torsion the 3-form field strength of the gravitational multiplet. The associated vector field is Killing and the 3-form is determined in terms of the geometry of spacetime. The $Sp(1)\ltimes \mathbb{H}$ case admits a descendant solution preserving 3 out of 4 supersymmetries due to the hyperini Killing spinor equation. If the isotropy group is compact, the spacetime admits a natural frame constructed from 1-form spinor bi-linears. In the $Sp(1)$ and $U(1)$ cases, the spacetime admits 3 and 4 parallel 1-forms with respect to the connection with torsion, respectively. The associated vector fields are Killing and under some additional restrictions the spacetime is a principal bundle with fibre a Lorentzian Lie group. The conditions imposed by the Killing spinor equations on all other fields are also determined.

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1 Introduction

In the past few years, there has been much work done to systematically solve the Killing spinor equations (KSEs) of supergravity theories and identify all solutions which preserve a fraction of spacetime supersymmetry. This programme, apart from its applications to supersymmetric theories, string theory and black holes, resembles the classification of instantons and monopoles of gauge theories. The difference is that the spacetime is now curved and there is a connection with special geometric structures on manifolds.

There are several supergravity theories in 6-dimensions. Here we shall be concerned with (1,0) supergravity, 8 real supercharges, coupled to tensor, vector and scalar multiplets. The theory has been constructed in [1, 2, 3]. The KSEs of 6-dimensional supergravities have been solved before in various special cases. In particular, the KSEs of minimal (1,0) supergravity have been solved in [4], and the maximally supersymmetric backgrounds have been classified in [4, 5]. The KSEs of (1,0) supergravity coupled to a tensor and some vector multiplets have been solved for backgrounds preserving one supersymmetry in [6]. The KSEs of (1,0) supergravity coupled a tensor, some vector and gauge multiplets have been solved for backgrounds preserving one supersymmetry in [7], see also [8]. Most of the computations carried out so far have been based on the method of spinor bi-linears [9] first applied to 5-dimensional supergravity. The only exception is the work of [5] where the integrability conditions of the KSEs were used as in [10].

In this paper, we shall solve the KSE of (1,0) supergravity coupled to any number of tensor, vector and scalar multiplets for backgrounds preserving any number of supersymmetries. For this, we shall use the spinorial geometry method of [11] and the apparent analogy that exists between the KSEs of (1,0) supergravity and those of heterotic supergravity. The latter have been solved in all generality [12, 13, 14]. We find that the solutions are characterized uniquely, apart from one case, by the isotropy group of the Killing spinors in $Spin(5,1) \cdot Sp(1)$. This is the holonomy of the supercovariant connection of a generic background. In particular, the isotropy groups of the spinors are

$$Sp(1) \cdot Sp(1) \ltimes \mathbb{H}(1) \ , \quad U(1) \cdot Sp(1) \ltimes \mathbb{H}(2) \ , \quad Sp(1) \ltimes \mathbb{H}(3,4) \ ;$$
$$Sp(1)(2) \ , \quad U(1)(4) \ , \quad \{1\}(8) \ , \tag{1.1}$$

where in parenthesis is the number of Killing spinors. Observe that in the $Sp(1) \ltimes \mathbb{H}$ case there is the possibility of a background to admit either 3 or 4 Killing spinors. To explain this, we note that in general only some of the solutions of the gravitino KSE to be also solutions of the other KSEs. Backgrounds for which the gravitino admits more solutions than the other KSEs are called descendants, see [13]. In the (1,0) supergravity, all backgrounds for which the gravitino KSE admits 4 or more solutions have descendants. However, after an analysis, we have shown that most of the descendants are not independent. This means that most of the descendant solutions are special cases of others for which all solutions of the gravitino KSE are also solutions of the other KSEs. The only case that this does not happen is that for the descendant $Sp(1) \ltimes \mathbb{H}$ backgrounds which preserve 3 supersymmetries. As we shall see, the conditions that arise from the hyperini KSE for 3 and 4 supersymmetries are different and so the $N = 3$ case gives rise to an independent descendant. The results on isotropy groups and the analysis for the descendants have been summarized in tables 1 and 2.

The geometry of the solutions depends on the isotropy group of the Killing spinors. There are two classes of solutions depending on whether the isotropy group is compact or non-compact. In the non-compact case and for backgrounds preserving one supersymmetry, the spacetime admits a parallel 1-form with respect to a metric connection, $\hat{\nabla}$, with skew-symmetric torsion, H, given by the 3-form field strength of the gravitational multiplet. As a result the spacetime admits a null Killing vector field. The 3-form field of the gravitational multiplet is completely determined in terms of the geometry of spacetime. In turn, the geometry of spacetime is characterized by the above mentioned parallel 1-form and a triplet of null 3-forms[2] which are constructed as Killing spinor bi-linears. The triplet of 3-forms in the directions transverse to the light-cone can be identified with the Hermitian self-dual forms in 4-dimensions. The 3-forms are also covariantly constant but this time with respect to a connection, $\mathcal{D}$, which apart from the skew-symmetric torsion part mentioned above, also includes an $Sp(1)$ connection which rotates the 3-forms. Such condition is similar to that of Quaternionic Kähler with torsion geometry [15]. The only difference is that the $Sp(1)$ connection may depend on the scalars of the hypermultiplet. In the $N = 2$ case, the spacetime admits the same form bi-linears, and so a null Killing vector field. The main difference is that one of the 3-form bi-linears is now parallel with respect to $\hat{\nabla}$. Though for the other two the covariant constancy conditions involves an additional $U(1)$ connection. Similarly in the $N = 4$ case, the spacetime admits the same form bi-linears. However all the 3-form bi-linears are now parallel with respect to $\hat{\nabla}$. The geometry of solutions with 3 supersymmetries is the same as that of backgrounds which preserve 4 supersymmetries. The difference is in the conditions that arise from the hyperini KSE.

In the compact case and for backgrounds preserving 2 supersymmetries, the spacetime admits 3 parallel 1-forms with respect to $\hat{\nabla}$. Therefore, the spacetime admits 3 isometries and H is determined in terms of these 1-forms and their first derivatives. The spacetime also admits 3 additional (vector bundle valued) 1-form bi-linears which now are parallel with respect to $\mathcal{D}$ connection. Therefore the co-tangent space of spacetime decomposes into a trivial rank 3 bundle spanned by the $\hat{\nabla}$-parallel 1-forms and the rest. Under some additional conditions, which are not implied by the KSEs, the spacetime can be thought as a principal bundle but in such a case it becomes a product $G \times \Sigma$, where G is locally $\mathbb{R}^{3,1}$ or $SL(2, \mathbb{R})$ and B is a 3-dimensional Riemannian manifold. The curvature of B is identified with that of an $Sp(1)$ connection which may be induced from the Quaternionic-Kähler manifold of scalar multiplets. Next for backgrounds which preserve 4 supersymmetries, the spacetime admits 4 $\hat{\nabla}$-parallel 1-form bi-linears. It also admits 2 (vector bundle valued) 1-form bi-linears which now are parallel with respect to $\mathcal{D}$ connection. Therefore the spacetime admits at least 4 isometries. The co-tangent spaces decomposes into a trivial rank 4 bundle spanned by the $\hat{\nabla}$-parallel 1-forms and the rest. As in the previous case, under some additional conditions which are not implied by the KSEs, the spacetime can be thought as a principal bundle. The fibre group has Lie algebra $\mathbb{R}^{3,1}$ or $\mathfrak{sl}(2, \mathbb{R}) \oplus \mathfrak{u}(1)$ or $\mathfrak{cw}_6$. However unlike the previous case, if the fibre group is not abelian, the spacetime is not a product. The curvature of the base space B is identified with that of a $U(1)$ connection which may be induced from the Quaternionic-Kähler manifold. In both compact and

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[2]These 3-forms are twisted with respect to an $Sp(1)$. So they should be thought of as a vector bundle valued 3-forms.

non-compact cases, the conditions imposed on the other fields from the KSEs have all been solved. In addition the fields have been expressed in terms of the geometry and their independent components.

This paper has been organized as follows. In section 2, we review the KSEs of 6-dimensional supergravity and explain their relation to those of heterotic supergravity. In section 3, we describe the solutions of the gravitino KSE and investigate the existence of descendants. In section 4, we present the geometry of backgrounds preserving 1 supersymmetry. In sections 5 and 6, we describe the geometry of backgrounds preserving 2 supersymmetries. Similarly in sections 7 and 8, we investigate the geometry of backgrounds preserving 4 supersymmetries as well as that of the $N = 3$ descendant. In section 9, we describe the backgrounds which preserve all 8 supersymmetries, and in section 10 we give our conclusions.

2 $(1,0)$ supergravity

2.1 Fields and KSEs

There are four types of (1,0)-supersymmetry multiplets in 6 dimensions, the graviton, tensor, vector and scalar mulptiplets. The bosonic fields of these multiplets are as follows: the graviton multiplet apart from the graviton has a 2-form gauge potential; the tensor multiplet has a 2-form gauge potential and a real scalar; the vector multiplet has a vector and the scalar multiplet has 4 (real) scalars. The theory we shall consider is (1,0)-supergravity coupled to n_T tensor, n_V vector and n_H scalar multiplets. The bosonic fields of the scalar multiplet, which is also referred as hypermultiplet, take values in a Quaternionic Kähler manifold which has real dimension $4n_H$.

Before we proceed to describe the KSEs, it is important to note that the fermions that appear in (1,0) supergravity satisfy a symplectic Majorana condition. This condition utilizes the invariant $Sp(1)$ and $Sp(n_H)$ forms to impose a reality condition of the spinors. Suppose that the Dirac or Weyl spinors λ and χ transform under the fundamental representations of $Sp(1)$ and $Sp(n_H)$, respectively. The symplectic Majorana condition is given by

$$\lambda^{\underline{A}} = \epsilon^{\underline{AB}} C \bar{\lambda}_{\underline{B}}^T , \quad \chi^{\underline{a}} = \epsilon^{\underline{ab}} C \bar{\chi}_{\underline{b}}^T ,\tag{2.1}$$

where C is the charge conjugation matrix and $\epsilon^{\underline{AB}}$ and $\epsilon^{\underline{ab}}$ are the symplectic invariant forms of $Sp(1)$ and $Sp(n_H)$, respectively, and $\underline{A}, \underline{B} = 1, 2$ and $\underline{a}, \underline{b} = 1, \ldots, 2n_H$.

We write the supersymmetry transformations of the fermions evaluated at the bosonic fields as

$$
\begin{aligned}
\delta\Psi_\mu^{\underline{A}} &= \nabla_\mu \epsilon^{\underline{A}} - \frac{1}{8} H_{\mu\nu\rho} \gamma^{\nu\rho} \epsilon^{\underline{A}} + \mathcal{C}_\mu{}^{\underline{A}}{}_{\underline{B}}\, \epsilon^{\underline{B}} , \\
\delta\chi^{M\underline{A}} &= \frac{i}{2} T_\mu^M \gamma^\mu \epsilon^{\underline{A}} - \frac{i}{24} H_{\mu\nu\rho}^M \gamma^{\mu\nu\rho} \epsilon^{\underline{A}} , \\
\delta\psi^{\underline{a}} &= i\gamma^\mu \epsilon_{\underline{A}} V_\mu^{\underline{a}\underline{A}} , \\
\delta\lambda^{a'\underline{A}} &= -\frac{1}{2\sqrt{2}} F_{\mu\nu}^{a'} \gamma^{\mu\nu} \epsilon^{\underline{A}} - \frac{1}{\sqrt{2}} (\mu^{a'})^{\underline{A}}{}_{\underline{B}} \epsilon^{\underline{B}} ,
\end{aligned}
\tag{2.2}
$$

3

where Ψ is the gravitino, χ is the tensorini, ψ is the hyperini and λ is the gaugini, ϵ is the superymmetry parameter and $a' = 1, \ldots, n_V$. The remaining coefficients that appear in the supersymmetry transformations depend on the fundamental fields of the theory. In turn, their explicit expressions depend on the formulation of the theory. The above structure of the superymmetry transformations that we have stated includes all known formulations. Most of the analysis on the solutions of the KSEs that follows is independent on the precise expression of supersymmetry transformations in terms of the fields. Because of this, we shall give the conditions that arise from the KSEs in generality. We shall also state explicitly where we use expression of the KSEs in terms of the fields. In what follows, we shall always assume that ∇ is the spin connection of the spacetime and $\mathcal{C}$ is a $Sp(1)$ connection.

To give an example of how the supersymmetry transformations,(2.2), depend on the fundamental fields of the theory, we shall mostly use the formulation[3] proposed in [3]. In this formulation, the organization of the fields is as follows. The theory has $n_T + 1$ 2-form gauge potentials $B^{\underline{r}}$, $r = 0, 1, \ldots, n_T$. One of the 2-form potentials is associated with the gravitational multiplet and the remaining n_T with the tensor multiplets. Let us denote the corresponding 3-form field strengths with $G^{\underline{r}}$. The precise relation between $B^{\underline{r}}$ and $G^{\underline{r}}$ will be given later as well as the duality conditions on $G^{\underline{r}}$. To continue, the scalar fields of the tensor multiplets parameterize the coset space $SO(1, n_T)/SO(n_T)$. A convenient way to describe this coset space is to choose a local section S as

$$S = \begin{pmatrix} v_{\underline{r}} \\ x_{\underline{r}}^{\underline{M}} \end{pmatrix} , \quad \underline{M} = 1, \ldots n_T \tag{2.3}$$

Since $S \in SO(1, n_T)$, one has $\tilde{S}\eta S = \eta$ where η is the Lorentz metric in $(1, n_T)$-dimensions. In particular

$$v_{\underline{r}} v^{\underline{r}} = 1 , \quad v_{\underline{r}} v_{\underline{s}} - \sum_{\underline{M}} x_{\underline{r}}^M x_{\underline{s}}^M = \eta_{\underline{rs}} , \quad v^{\underline{r}} x_{\underline{r}}^M = 0 . \tag{2.4}$$

The canonical $SO(n_T)$ connection of the coset is $\sum_{\underline{r}} x_{\underline{r}}^M dx_{\underline{r}}^N$.

The scalars of the hypermultiplet parameterize a Quaternionic Kähler manifold which has holonomy $Sp(n_H) \cdot Sp(1)$. Such a manifold admits a frame E such that the metric can be written as

$$g_{IJ} = E_{\underline{I}}^{aA} E_{\underline{J}}^{bB} \epsilon_{\underline{ab}} \epsilon_{\underline{AB}} , \tag{2.5}$$

where $\epsilon_{\underline{ab}}$ and $\epsilon_{\underline{AB}}$ are the invariant $Sp(n_H)$ and $Sp(1)$ 2-forms, respectively. The associated spin connection has holonomy $Sp(n_H) \cdot Sp(1)$ and so decomposes as $(\mathcal{A}_{\underline{I}\underline{b}}^a, \mathcal{A}_{\underline{I}B}^A)$.

In [3] to include vector multiplets with (non-abelian) gauge potential $A_\mu^{a'}$, one assumes that the quaternionic Kähler manifold[4] of the hypermultiplet is $Sp(1, n_H)/Sp(1) \times Sp(n_H)$ and gauges the maximal compact isometry subgroup $Sp(1) \times Sp(n_H)$. So the gauge group

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[3]We use a different normalization for some of the fields from that in [3]. Our normalization is similar to that of heterotic supergravity.

[4]It is likely that this assumption is not necessary and a more general class of models can exist. Moreover μ may be related to moment maps [16] of Quaternionic Kähler geometry.

of the theory is $H = Sp(1) \times Sp(n_H) \times K$, where K is a product of semi-simple groups which does not act on the scalars. Let $\xi_{a'_1}$ and $\xi_{a'_2}$ be the vector fields generated on $Sp(1, n_H)/Sp(1) \times Sp(n_H)$ by the action of $Sp(1)$ and $Sp(n_H)$, respectively. Under these assumptions, one has that

$$
\begin{aligned}
&H_{\mu\nu\rho} = v_{\underline{r}} G^{\underline{r}}_{\mu\nu\rho} \,, \quad H^M_{\mu\nu\rho} = x^M_{\underline{r}} G^{\underline{r}}_{\mu\nu\rho} \,, \quad \mathcal{C}_\mu{}^A{}_{\underline{B}} = D_\mu \phi^{\underline{I}} \mathcal{A}_{\underline{I}}{}^A{}_{\underline{B}} \,, \\
&T^M_\mu = x^M_{\underline{r}} \partial_\mu v^{\underline{r}} \,, \quad V^{\underline{a}A}_\mu = E^{\underline{a}A}_{\underline{I}} D_\mu \phi^{\underline{I}} \,, \quad F^{a'}_{\mu\nu} = \partial_\mu A^{a'}_\nu - \partial_\nu A^{a'}_\mu + f^{a'}{}_{b'c'} A^{b'}_\mu A^{c'}_\nu \,, \\
&(\mu^{a'_1})^A{}_{\underline{B}} = -\frac{2}{v_{\underline{r}} c^{\underline{r}1}} \mathcal{A}_{\underline{I}}{}^A{}_{\underline{B}} \xi^{I a'_1} \,, \quad (\mu^{a'_2})^A{}_{\underline{B}} = -\frac{2}{v_{\underline{r}} c^{\underline{r}2}} \mathcal{A}_{\underline{I}}{}^A{}_{\underline{B}} \xi^{I a'_2} \,, \quad (\mu^{a'_3})^A{}_{\underline{B}} = 0 \,, (2.6)
\end{aligned}
$$

where the gauge index a'_3 ranges over the gauge subgroup K, $\phi^{\underline{I}}$ are the scalars of the hypermultiplet,

$$
\begin{aligned}
\nabla_\mu \epsilon^{\underline{A}} &= \partial_\mu \epsilon^{\underline{A}} + \frac{1}{4} \Omega_{\mu,mn} \gamma^{mn} \epsilon^{\underline{A}} \,, \\
D_\mu \phi^{\underline{I}} &= \partial_\mu \phi^{\underline{I}} - A^{a'}_\mu \xi^{\underline{I}}_{a'} \,,
\end{aligned} \tag{2.7}
$$

respectively, and Ω is the frame connection of spacetime. It is understood that $\xi_{a'_3} = 0$ as K does not act on the scalars of the hypermultiplet. Clearly $F^{a'}$ are the field strengths of the gauge potentials $A^{a'}$ and f are the structure constants of the gauge group H.

It remains to define the field strengths $G^{\underline{r}}$. These are given by

$$
G^{\underline{r}}_{\mu\nu\rho} = 3 \partial_{[\mu} B^{\underline{r}}_{\nu\rho]} + c^{\underline{r}1} CS(A^{Sp(1)})_{\mu\nu\rho} + c^{\underline{r}2} CS(A^{Sp(n_H)})_{\mu\nu\rho} + c^{\underline{r}K} CS(A^K)_{\mu\nu\rho} \,, \tag{2.8}
$$

where $c^{\underline{r}}$'s are constants, one for each copy of the gauge group, and $CS(A)$'s are the Chern-Simons 3-forms. Observe that the constants $c^{\underline{r}1}$ and $c^{\underline{r}2}$ enter in the definition of μ's in (2.6).

The duality condition on G is given by

$$
\zeta_{\underline{rs}} G^{\underline{s}}_{\mu_1\mu_2\mu_3} = \frac{1}{3!} \epsilon_{\mu_1\mu_2\mu_3}{}^{\nu_1\nu_2\nu_3} G_{\underline{r}\nu_1\nu_2\nu_3} \,, \tag{2.9}
$$

where

$$
\zeta_{\underline{rs}} = v_{\underline{r}} v_{\underline{s}} + \sum_M x^M_{\underline{r}} x^M_{\underline{s}} \,. \tag{2.10}
$$

Note that the duality conditions for H and H^M are opposite. In our conventions, H is anti-self-dual while H^M is self-dual.

2.2 Spinors

The spinorial geometry technique to solve the Killing spinor equations is applied most effectively provided we express the spinors in terms of forms. In particular, we have to find a way to impose the symplectic Majorana condition on the spinors. For this we identify the symplectic Majorana-Weyl $Spin(5,1)$ spinors with $SU(2)$-invariant Majorana-Weyl $Spin(9,1)$ spinors. Under this identification the symplectic-Majorana condition on the $Spin(5,1)$ spinors is replaced by the Majorana condition on the $Spin(9,1)$ spinors. To do

this explicitly, recall that the Dirac spinors of $Spin(9,1)$ are identified with $\Lambda^*(\mathbb{C}^5)$, and the positive and negative chirality spinors are the even and odd degree forms, respectively. The gamma matrices of $\text{Clif}(\mathbb{R}^{9,1})$ are given by

$$
\begin{aligned}
\Gamma_0 &= -e_5 \wedge + e_5 \lrcorner \,, \quad \Gamma_5 = e_5 \wedge + e_5 \lrcorner \,, \\
\Gamma_i &= e_i \wedge + e_i \lrcorner \,, \quad \Gamma_{i+5} = i(e_5 \wedge - e_5 \lrcorner) \,, \quad i = 1, 2, 3, 4 \,,
\end{aligned}
\tag{2.11}
$$

where e_i, $i = 1, \ldots, 5$, is a Hermitian basis in $\mathbb{C}^5$. The gamma matrices of $\text{Clif}(\mathbb{R}^{5,1})$ are identified as

$$
\gamma_\mu = \Gamma_\mu \,, \quad \mu = 0, 1, 2 \,; \quad \gamma_\mu = \Gamma_{\mu+2} \,, \quad \mu = 3, 4, 5 \,.
\tag{2.12}
$$

Therefore the positive chirality Weyl spinors of $Spin(5,1) = SL(2, \mathbb{H})$ are $\Lambda^{\text{ev}}(\mathbb{C}\langle e_1, e_2, e_5\rangle) = \mathbb{H}^2$. The symplectic Majorana-Weyl condition of $Spin(5,1)$ is the Majorana-Weyl condition of $Spin(9,1)$ spinors, ie

$$
\epsilon^* = \Gamma_{67}\Gamma_{89}\epsilon \,,
\tag{2.13}
$$

where $\epsilon \in \Lambda^{\text{ev}}\mathbb{C}\langle e_1, e_2, e_5\rangle \otimes \Lambda^*\mathbb{C}\langle e_{34}\rangle$. In particular a basis for the symplectic Majorana-Weyl spinors is

$$
\begin{aligned}
&1 + e_{1234} \,, \quad i(1 - e_{1234}) \,, \quad e_{12} - e_{34} \,, \quad i(e_{12} + e_{34}) \,, \\
&e_{15} + e_{2534} \,, \quad i(e_{15} - e_{2534}) \,, \quad e_{25} - e_{1534} \,, \quad i(e_{25} + e_{1534}) \,.
\end{aligned}
\tag{2.14}
$$

Observe that the above basis selects the diagonal of two copies of the Weyl representation of $Spin(5,1)$, where the first copy is $\Lambda^{\text{ev}}(\mathbb{C}\langle e_1, e_2, e_5\rangle)$ while the second copy includes the auxiliary direction e_{34}. The $SU(2)$ acting on the auxiliary directions e_3 and e_4 leaves the basis invariant.

2.3 KSEs revisited

It remains to rewrite the KSEs of 6-dimensional supergravity in terms of the 10-dimensional notation we have introduced above. For this, we define $\rho^{r'}$, $r' = 1, 2, 3$, such that

$$
\rho^1 = \frac{1}{2}(\Gamma_{38} + \Gamma_{49}) \,, \quad \rho^2 = \frac{1}{2}(\Gamma_{89} - \Gamma_{34}) \,, \quad \rho^3 = \frac{1}{2}(\Gamma_{39} - \Gamma_{48}) \,.
\tag{2.15}
$$

Observe that these are the generators of the Lie algebra $Sp(1)$ as it acts on the basis (2.14). Using this the KSEs can be rewritten as

$$
\begin{aligned}
\mathcal{D}\epsilon \equiv \left(\nabla_\mu - \frac{1}{8}H_{\mu\nu\rho}\gamma^{\nu\rho} + \mathcal{C}_\mu^{r'}\rho_{r'}\right)\epsilon &= 0, \\
\left(\frac{i}{2}T_\mu^M \gamma^\mu - \frac{i}{24}H_{\mu\nu\rho}^M \gamma^{\mu\nu\rho}\right)\epsilon &= 0, \\
i\gamma^\mu \epsilon_{\underline{A}} V_\mu^{\underline{aA}} &= 0 \,, \\
\left(\frac{1}{4}F_{\mu\nu}^{a'}\gamma^{\mu\nu} + \frac{1}{2}\mu_{r'}^{a'}\rho^{r'}\right)\epsilon &= 0 \,.
\end{aligned}
\tag{2.16}
$$

In the hyperini KSE, it should be understood that

$$
\epsilon_1 = -\epsilon^2 \,, \quad \epsilon_2 = \Gamma_{34}\epsilon^1 \,,
\tag{2.17}
$$

where ϵ^1 and ϵ^2 are the components of ϵ in the two copies of the Weyl representation used to construct the symplectic-Majorana representation.

3 Parallel and Killing spinors

3.1 Parallel spinors

The (reduced) holonomy[5] of 6-dimensional supergravity supercovariant connection $\mathcal{D}$, (2.16), is contained in $Spin(5,1) \cdot Sp(1)$. This is the same as the gauge group of the theory. Therefore there are two possibilities. Either the parallel spinors have a trivial isotropy group in $Spin(5,1) \cdot Sp(1)$ or the parallel spinors have a non-trivial isotropy group in $Spin(5,1) \cdot Sp(1)$. To investigate the two cases, consider the integrability of the gravitino Killing spinor equation which gives

$$\frac{1}{4}\hat{R}_{\mu\nu,\rho\sigma}\gamma^{\rho\sigma}\epsilon + \mathcal{F}_{\mu\nu}^{r'}\rho_{r'}\epsilon = 0 \ , \tag{3.1}$$

where

$$\mathcal{F}_{\mu\nu}^{r'} = \partial_\mu \mathcal{C}_\nu^{r'} - \partial_\nu \mathcal{C}_\mu^{r'} + 2\epsilon^{r'}{}_{s't'}\mathcal{C}_\mu^{s'}\mathcal{C}_\nu^{t'} - H^\lambda{}_{\mu\nu}\mathcal{C}_\lambda^{r'} \ , \tag{3.2}$$

and $\hat{R}$ is the curvature of the connection, $\hat{\nabla}$, with skew-symmetric torsion H defined as

$$\hat{\nabla}_\mu Y^\nu = \nabla_\mu Y^\nu + \frac{1}{2}H^\nu{}_{\mu\lambda}Y^\lambda \ . \tag{3.3}$$

3.1.1 Trivial isotropy group

Now if the isotropy group of the parallel spinors is $\{1\}$, a direct inspection of (3.1) reveals that

$$\hat{R} = 0 \ , \quad \mathcal{F} = 0 \ . \tag{3.4}$$

The spacetime is parallelizable with respect to a connection with skew-symmetric torsion and admits 8 parallel spinors. Moreover, the torsion is anti-self-dual. All such spacetimes are group manifolds with anti-self-dual structure constants.

3.1.2 Non-trivial isotropy group

Next suppose that the parallel spinors have a non-trivial isotropy group in $Spin(5,1) \cdot Sp(1)$. To find the isotropy groups, we first remark that $Spin(5,1) = SL(2,\mathbb{H})$ and the action of $Spin(5,1) \cdot Sp(1)$ on the symplectic Majorana-Weyl spinors can be described in terms of quaternions. In particular, the symplectic Majorana-Weyl spinors can be identified with $\mathbb{H}^2$ with $Spin(5,1) = SL(2,\mathbb{H})$ acting from the left with quaternionic matrix multiplication while $Sp(1)$ acts on the right with the conjugate quaternionic multiplication. Using, this it is easy to see that there is a single non-trivial orbit of $Spin(5,1) \cdot Sp(1)$ on the symplectic Majorana-Weyl spinors with isotropy group $Sp(1) \cdot Sp(1) \ltimes \mathbb{H}$. To continue, we have to determine the action of $Sp(1) \cdot Sp(1) \ltimes \mathbb{H}$ on $\mathbb{H}^2$. Decomposing

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[5]We assume that the backgrounds are simply connected or equivalently we consider the universal cover.

$\mathbb{H}^2 = \mathbb{R} \oplus \mathrm{Im}\mathbb{H} \oplus \mathbb{H}$, where $\mathbb{R}$ is chosen to be along the first invariant spinor, then the action of the isotropy group is

$$\mathrm{Im}\mathbb{H} \oplus \mathbb{H} \to a\mathrm{Im}\mathbb{H}\bar{a} \oplus b\mathbb{H}\bar{a} \ , \qquad\qquad (3.5)$$

where $(a, b) \in Sp(1) \cdot Sp(1)$ and $\bar{a}$ is the quaternionic conjugate of $a \in Sp(1)$. There are two possibilities. Either the second invariant spinor lies in $\mathrm{Im}\mathbb{H}$ or in $\mathbb{H}$. It cannot lie in both because if there is a non-trivial component in $\mathbb{H}$, there is a $\mathbb{H}$ transformation in $Sp(1) \cdot Sp(1) \ltimes \mathbb{H}$ such that the component in $\mathrm{Im}\mathbb{H}$ can be set to zero. Now if the second spinor lies in $\mathrm{Im}\mathbb{H}$, the isotropy group is $U(1) \cdot Sp(1) \ltimes \mathbb{H}$. On the other hand if it lies in $\mathbb{H}$, the isotropy group is $Sp(1)$. This concludes the analysis for two invariant spinors.

To continue, it is easy to see that if there are 3 invariant spinors, then there always exist an additional one. For 4 invariant spinors, there are two cases to consider with non-trivial isotropy group. Either all four invariant spinors span the first copy of $\mathbb{H}$ in $\mathbb{H}^2$ and the isotropy group is $Sp(1) \ltimes \mathbb{H}$, or 2 lie in the first copy and the other 2 lie in the second copy of $\mathbb{H}$ in $\mathbb{H}^2$ and the isotropy group is $U(1)$. The isotropy group of more than 4 linearly independent spinors is $\{1\}$. The above results as well as representatives of the invariant spinors have been summarized in table 1.

N	Isotropy Groups	Spinors
1	$Sp(1) \cdot Sp(1) \ltimes \mathbb{H}$	$1 + e_{1234}$
2	$(U(1) \cdot Sp(1)) \ltimes \mathbb{H}$	$1 + e_{1234} \ , \ i(1 - e_{1234})$
4	$Sp(1) \ltimes \mathbb{H}$	$1 + e_{1234} \ , \ i(1 - e_{1234}) \ , \ e_{12} - e_{34} \ , \ i(e_{12} + e_{34})$
2	$Sp(1))$	$1 + e_{1234} \ , \ e_{15} + e_{2345}$
4	$U(1)$	$1 + e_{1234} \ , \ i(1 - e_{1234}) \ , \ e_{15} + e_{2345} \ , \ i(e_{15} - e_{2345})$

Table 1: The first column gives the number of invariant spinors, the second column the associated isotropy groups and the third representatives of the invariant spinors. Observe that if 3 spinors are invariant, then there is a fourth one. Moreover the isotropy group of more than 4 spinors is the identity.

3.2 Descendants

A distinguished class of supersymmetric backgrounds are those for which all parallel spinors given in table 1 are Killing, ie they solve all KSEs. However, it is not always the case that all solutions of the gravitino KSE are also solutions of the other three KSEs. Typically, only some or linear combinations of the parallel spinors are Killing. This is similar to the heterotic case where an extensive analysis was required to identify the "descendant" solutions [13], ie the solutions that had less Killing than parallel spinors. However unlike the heterotic, the analysis required to identify the descendants backgrounds of 6-dimensional supergravity is simpler. As we shall see there are many descendants but in most cases the Killing spinors of descendants are given in terms of the parallel spinors of table 1. Such descendant backgrounds are special cases of solutions for which all parallel spinors are Killing. The objective of the analysis which follows is to find

whether there are backgrounds which have Killing spinors that differ from those given in table 1. If they exist, such backgrounds will be called independent descendant solutions or simply "independent".

In all cases, if a solution has just one Killing spinor, irrespective of the number of parallel spinors, it is always possible to rotate it so that it is identified with $1 + e_{1234}$. Therefore such descendant backgrounds are included in those for which $1 + e_{1234}$ is both parallel and Killing spinor and so they are not independent. Using this, the cases we have to examine are those with two or more Killing and with four or more parallel spinors.

3.3 Descendants of four parallel spinors

3.3.1 $Sp(1) \ltimes \mathbb{H}$

Suppose that a solution has 4 parallel but only 2 Killing spinors. There are two cases to consider depending on the isotropy group of the parallel spinors. If the isotropy group of the parallel spinors is $Sp(1) \ltimes \mathbb{H}$, then the sigma group [13] is $Spin(1,1) \times Sp(1) \cdot Sp(1)$. The subgroup $Sp(1) \cdot Sp(1) = SO(4)$ acts with the vector representation on the 4 parallel spinors. In such a case, it is always possible to arrange such that the first two Killing spinors are

$$1 + e_{1234} \ , \ i(1 - e_{1234}) \ . \tag{3.6}$$

Therefore such solutions are special cases of backgrounds with 2 supersymmetries associated with 2 parallel spinors with isotropy group $U(1) \cdot Sp(1)) \ltimes \mathbb{H}$, and so they are not independent.

Next suppose that a solution has 3 Killing spinors. Again since the subgroup $Sp(1) \cdot Sp(1)$ of the sigma group acts with the the vector representation, it is always possible to choose the 3 Killing spinors as

$$1 + e_{1234} \ , \ i(1 - e_{1234}) \ , \quad e_{12} - e_{34} \ . \tag{3.7}$$

It turns out that if the gravitino, tensorini and gaugini KSE admit (3.7) as a solution, then they admit also $i(e_{12} + e_{34})$ as a solution. Thus all the parallel spinors of this case solve the three out of four KSEs. It remains to investigate the hyperini KSE. We shall see that the conditions that arise from the hyperini KSE evaluated on (3.7) are different from those that one finds when the same KSE is evaluated on all 4 $Sp(1) \ltimes \mathbb{H}$-invariant spinors. As a result, the KSEs allow for backgrounds with 3 supersymmetries. However the existence of such backgrounds depends also on the field equations.

3.3.2 $U(1)$

It remains to investigate the case for which the 4 parallel spinors have isotropy group $U(1)$. The sigma group [13] in this case is $Spin(3,1) \times U(1)$. One way to see this is to treat the directions 2,3 and 4 in the $U(1)$-invariant spinors given in table 1 as auxiliary and suppress them. Then the spinors can be identified with the Majorana spinors of $Spin(3,1)$. The $U(1)$ subgroup of the sigma group is generated by spin transformations along the auxiliary directions. The analysis of the orbits of the sigma group is identical to

that of the gauge group of 4-dimensional supergravity [17]. Thus there are two different cases of descendants with 2 supersymmetries that we must consider. Using in addition the $U(1)$ subgroup of the sigma group, one can arrange such that the Killing spinors of the two cases are identical to the parallel spinors of table 1 with isotropy groups $U(1) \cdot Sp(1) \ltimes \mathbb{H}$ and $Sp(1)$, respectively. Therefore both cases are special cases of other backgrounds with less parallel spinors, and so they are not independent.

Next consider the case of backgrounds with 3 Killing spinors. The existence of such backgrounds depends on the details of the Killing spinor equations. To see whether such solution can exist, one can pick the 3-plane of Killing spinors by using the sigma group to bring the normal spinor of the 3-plane to a canonical form. The procedure is explained in detail in [18, 13]. It turns out that the normal spinor can be chosen such that the 3 Killing spinors lie on the 3-plane spanned by

$$1 + e_{1234} \ , \ i(1 - e_{1234}) \ , \ e_{15} + e_{2345} \ . \tag{3.8}$$

It is easy to see that if (3.8) solve the gravitino, tensorini and gaugini KSEs, then $i(e_{15} - e_{2345})$ is also a solution. As a result all 4 $U(1)$-invariant spinors are solutions to these three KSEs. It remains to examine the hyperini KSE. Unlike the previous case, the hyperini KSE evaluated on (3.8) gives the same conditions as those one obtains for all 4 $U(1)$-invariant spinors. Thus in this case there are no descendants preserving 3 supersymmetries.

$\mathrm{hol}(\mathcal{D})$	N
$Sp(1) \cdot Sp(1) \ltimes \mathbb{H}$	1
$U(1) \cdot Sp(1) \ltimes \mathbb{H}$	$*$, 2
$Sp(1) \ltimes \mathbb{H}$	$*$, $*$, 3, 4
$Sp(1)$	$*$, 2
$U(1)$	$*$, $*$, $-$, 4
$\{1\}$	$*$,$*$,$*$, $*$,$-$, $-$, $-$, 8

Table 2: In the columns are the holonomy groups that arise from the solution of the gravitino KSE and the number N of supersymmetries, respectively. $*$ entries denote the cases that occur but are special cases of others with the same number of supersymmetries but with less parallel spinors. The $-$ entries denote cases which do not occur. The Killing spinors for $N = 1, 2, 4$ are the same as those given in table 1 while for $N = 3$ in (3.7).

3.4 Descendants of eight parallel spinors

It remains to examine the descendants of backgrounds with 8 parallel spinors. For this it is convenient to solve the KSEs in the order

$$\mathrm{gravitino} \to \mathrm{gaugini} \to \mathrm{tensorini} \to \mathrm{hyperini} \ . \tag{3.9}$$

We have already stated that the gravitino KSE admits 8 parallel spinors. It remains to investigate the remaining three KSEs.

3.4.1 Gaugini

The solutions of the gaugini KSE are spinors which are invariant under some subgroup of $Spin(5,1) \cdot Sp(1)$. This is because the gauge field and moment maps can be viewed as maps from $\mathfrak{spin}(5,1) \oplus \mathfrak{sp}(1)$ to the Lie algebra of the gauge group, where $\mathfrak{spin}(5,1) = \Lambda^2(\mathbb{R}^{5,1})$. But all such spinors and their isotropy groups have been tabulated in table 1. Thus the gaugini KSE can preserve 1, 2(2), 4(2) and 8 out of the total of 8 parallel spinors, where the number in the parenthesis states the multiplicity of each case.

Having established that the gaugino KSE has solutions given by the spinors of table 1, it remains to investigate the remaining two KSEs. If the gaugini KSE has up to 4 solutions, the investigation of the descendants for the tensorini and hyperini KSEs is the same as that presented in section 3.3. In particular, there is one descendant with 3 supersymmetries which arises in the case of 4 Killing spinors with isotropy group $Sp(1) \ltimes \mathbb{H}$. The three Killing spinors are given in (3.7). So this case can be thought as a special case of backgrounds with 4 parallel spinors and Killing spinors given in (3.7). Since we have dealt with all descendants of the gaugini KSE from now on we shall take that the gaugini KSE preserves all 8 parallel spinors.

3.4.2 Tensorini

Let us assume that the gravitino and gaugini KSEs admit 8 Killing spinors. Observe that the tensorini KSE commutes with all 3 ρ operations given in (2.15). Because of this it preserves either 4 or 8 supersymmetries. Moreover, whenever it preserves 4 supersymmetries, the Killing spinors can be given in terms of the $Sp(1) \ltimes \mathbb{H}$-invariant spinors of table 1. Using this, one can solve the hyperini KSE to find that the backgrounds preserve 1,2,3 and 4 supersymmetries. All of them are special cases of solutions which we have already investigated. In particular, if the solutions preserves one supersymmetry, then it is a special case of backgrounds with one parallel spinor which is also Killing. In the N=2 case, the backgrounds are special cases of solutions with two parallel spinors which are also Killing and have isotropy group $Sp(1) \cdot U(1) \ltimes \mathbb{H}$. For $N = 3$, the backgrounds are special cases of those with $Sp(1) \ltimes \mathbb{H}$-invariant parallel spinors and 3 Killing spinors given in (3.7). The $N = 4$ case is included in that for which the 4 $Sp(1) \ltimes \mathbb{H}$-invariant parallel spinors are also Killing. This concludes the analysis of the descendants in this case, so from now one we shall assume that the tensorini KSE admits 8 Killing spinors.

3.4.3 Hypernini

Let us assume that the gravitino, gaugini and tensorini KSEs admit 8 Killing spinors. To investigate solutions of the hyperini KSE, we have to identify the orbits of the sigma group, which in this case is $Spin(5,1) \cdot Sp(1)$, on the space of spinors. We have already dealt with the descendants preserving one supersymmetry. The Killing spinor can be identified with $1 + e_{1234}$. To investigate the case with 2 supersymmetries, we first recall that the sigma group $Spin(5,1) \cdot Sp(1)$ has one orbit in the space of symplectic-Majorana spinors with isotropy group $Sp(1) \cdot Sp(1) \ltimes \mathbb{H}$. The representative can be chosen as $1 + e_{1234}$. The action of the isotropy group on the space of spinors is given in (3.5). This isotropy group has two non-trivial orbits on the space of spinors and the representatives can be chosen

as either $i(1 - e_{1234})$ or $e_{15} + e_{2345}$. It is clear from this that solutions with Killing spinors $1 + e_{1234}$ and $i(1 - e_{1234})$ or $1 + e_{1234}$ and $e_{15} + e_{2345}$ are not independent descendants. So there no independent descendants with two supersymmetries.

Next let us consider the case with 3 supersymmetries. There are two cases to investigate. First suppose that the isotropy group of the first two spinors is $Sp(1) \cdot U(1) \ltimes \mathbb{H}$. This group has two different orbits on the rest of the spinors with representatives $e_{12} - e_{34}$ and $e_{15} + e_{2345}$, respectively. These two cases are not new as the Killing spinors are identical to those found in (3.7) and (3.8), respectively. In addition one can show that if the hyperini KSE admits (3.8) as Killing spinors, then it preserves 4 supersymmetries with Killing spinors as the $U(1)$-invariant spinors of table 1.

Next suppose that the isotropy group of the first two Killing spinors is $Sp(1)$. It can be easily seen from (3.5) that $Sp(1)$ acts with two copies of the 3-dimensional representation on the remaining 6 spinors. As a result it can be arranged such that the third spinor can be chosen in such a way that the three Killing spinors are

$$1 + e_{1234} , \quad e_{15} + e_{2345} , \quad c_1 i(1 - e_{1234}) + i c_2(e_{15} - e_{2345}) + c_3(e_{25} - e_{1345}) , \quad (3.10)$$

where c's are constants. If $c_1 = 0$, then the third spinor can be simplified further by choosing $c_3 = 0$. As we shall see, there are no new descendants. The hyperini KSE evaluated on the above spinors implies that either it preserves four supersymmetries with Killing spinors as the $U(1)$-invariant spinors of table 1 or it preserves all 8 supersymmetries. This depends on the coefficients c.

It remains to investigate descendants with 4 supersymmetries. First suppose that the first 3 Killing spinors are chosen as in (3.7). The isotropy group in this case is $Sp(1) \ltimes \mathbb{H}$. This has two orbits on the remaining spinors. The representatives can be chosen such that the four Killing spinors are given by either the 4 $Sp(1) \ltimes \mathbb{H}$-invariant spinors of table 1 or

$$1 + e_{1234} , \quad i(1 - e_{1234}) , \quad e_{12} - e_{34} , \quad e_{15} + e_{2345} . \quad (3.11)$$

This can be a new descendant. However it turns out that if the hyperini KSE preserves the above 4 spinors, then it preserves all 8 supersymmetries.

Next suppose that the first 3 Killing spinors are given in (3.8). The isotropy group of these spinors is $U(1)$. Thus the fourth spinor can be chosen as

$$c_1(e_{12} - e_{34}) + c_2 i(e_{15} - e_{2345}) + c_3(e_{25} - e_{1345}) + c_4 i(e_{25} + e_{1345}) . \quad (3.12)$$

It turns out depending on the choice of the coefficients c that the hyperini KSE preserves either 4 supersymmetries with Killing spinors given by the $U(1)$-invariant spinors of table 1 or all 8 supersymmetries. So there are no new descendants. A similar conclusion holds for the case for which the third Killing spinor is chosen as in (3.10).

To conclude, if the isotropy group of parallel spinors is $\{1\}$, there are descendant backgrounds which preserve 1, 2, 3 and 4 supersymmetries. However they are not independent. All of them are special cases of backgrounds that admit less parallel spinors. The results for all descendants have been tabulated in table 2.

4   N=1

The lexicographic structure of 6-dimensional supergravity KSEs is similar to that of heterotic supergravity. As a result, the results of [12, 13] can be adapted to 6-dimensions. Because of this, we shall not explain the calculations in detail. The only difference is in the hyperini KSE which is examined separately.

4.1   Gravitino

As the gauge group of the theory is the same as the holonomy of supercovariant connection of generic backgrounds, the Killing spinor of $N = 1$ backgrounds can be chosen as $\epsilon = 1 + e_{1234}$, see [12, 13] for an explanation. The gravitino KSE requires that this spinor is parallel. As a result the holonomy of $\mathcal{D}$ reduces to a subgroup of the isotropy group $Sp(1) \cdot Sp(1) \ltimes \mathbb{H}$ of the parallel spinor, ie

$$\mathrm{hol}(\mathcal{D}) \subseteq Sp(1) \cdot Sp(1) \ltimes \mathbb{H} \ . \tag{4.1}$$

This is the full content of the gravitino KSE. The restrictions that this imposes on the geometry will be examined later.

4.2   Gaugini

A direct application of the spinorial geometry technique [11] reveals that the conditions that arise from the gaugini KSE are

$$F^{a'}_{+i} = F^{a'}_{+-} = 0 \ , \quad F^{a'\,\alpha}_{\alpha} + i\mu^1 = 0 \ , \quad 2F^{a'}_{12} + \mu^2 - i\mu^3 = 0 \ . \tag{4.2}$$

It is clear that the gauge field strength vanishes along one of the light-cone directions.

4.3   Tensorini

A direct computation of the tensorini KSE on the spinor $1 + e_{1234}$, or a comparison with the solution of the dilatino KSE for heterotic backgrounds preserving one supersymmetry, reveals that

$$T^{\underline{M}}_{+} = 0 \ , \quad H^{\underline{M}}_{+\alpha}{}^{\alpha} = H^{\underline{M}}_{+\alpha\beta} = 0 \ ,$$
$$T^{\underline{M}}_{\bar{\alpha}} - \frac{1}{2}H^{\underline{M}}_{-+\bar{\alpha}} - \frac{1}{2}H^{\underline{M}}_{\bar{\alpha}\beta}{}^{\beta} = 0 \ . \tag{4.3}$$

Note that the tensorini KSE commutes with the Clifford algebra operations $\rho^{r'}$ in (2.15). As a result, if the tensorini KSE admits a solution ϵ, then $\rho^{r'}\epsilon$ also solve the KSE. As a result, the four spinors

$$1 + e_{1234} \ , \quad \rho^{r'}(1 + e_{1234}) \ , \quad r' = 1, 2, 3, \tag{4.4}$$

are solutions to the tensorini KSE.

4.4 Hyperini

To understand the hyperini KSE, one has to identify the ϵ_A components of the Killing spinor in the context of spinorial geometry. In our notation $\epsilon^1 = 1$ and $\epsilon^2 = e_{1234}$ and since $\epsilon_1 = -\epsilon^2$ and $\epsilon_2 = \Gamma_{34}\epsilon^1$, one has $\epsilon_1 = -e_{1234}$ and $\epsilon_2 = e_{34}$. Substituting these into the KSE, one finds the conditions

$$V_+^{aA} = 0 \ , \quad -V_1^{a1} + V_{\bar{2}}^{a2} = 0 \ , \quad V_2^{a1} + V_{\bar{1}}^{a2} = 0 \ . \tag{4.5}$$

Expressing the coefficients of the KSEs in terms of the fundamental fields as in (2.6), it is clear that

$$D_+\phi^I = 0 \ . \tag{4.6}$$

4.5 Geometry

4.5.1 Form spinor bi-linears

To investigate further the geometry of spacetime, one has to compute the form spinor bi-linears. The form spinor bi-linears of two spinors are given by

$$\tau = \frac{1}{k!} B(\epsilon_1, \gamma_{\mu_1 \ldots \mu_k} \epsilon_2) \ e^{\mu_1} \wedge \cdots \wedge e^{\mu_k} \ , \tag{4.7}$$

where B is the Majorana inner product as for the heterotic supergravity. Assuming that ϵ_1 and ϵ_2 satisfy the gravitino KSE, it is easy to see that

$$\hat{\nabla}_\nu \tau = 0 \ . \tag{4.8}$$

The form τ is covariantly constant with respect to $\hat{\nabla}$ and the $Sp(1)$ connection $\mathcal{C}^{r'}$ does not contribute in the parallel transport equation.

On the other hand, one may also consider the $\mathfrak{sp}(1)$-valued form bi-linears

$$\tau^{r'} = \frac{1}{k!} B(\epsilon_1, \gamma_{\mu_1 \ldots \mu_k} \rho^{r'} \epsilon_2) \ e^{\mu_1} \wedge \cdots \wedge e^{\mu_k} \ . \tag{4.9}$$

Assuming again that ϵ_1 and ϵ_2 satisfy the gravitino KSE, one finds that

$$\hat{\nabla}_\nu \tau^{r'} + 2\mathcal{C}_\nu^{s'} \epsilon^{r'}{}_{s't'} \tau^{t'} = 0 \ . \tag{4.10}$$

Observe that the $\mathfrak{sp}(1)$-valued form bi-linears are twisted with respect to the $Sp(1)$ connection $\mathcal{C}^{r'}$. So $\nabla_\nu \tau^{r'}$ are not forms but rather vector bundle valued forms. However for simplicity in what follows, we shall refer to both τ and $\tau^{r'}$ as forms.

4.5.2 Spacetime geometry of N=1 backgrounds

The algebraic independent bi-linears of backgrounds preserving one supersymmetry are

$$e^- \ , \quad e^- \wedge \omega_I \ , \quad e^- \wedge \omega_J \ , \quad e^- \wedge \omega_K \ , \tag{4.11}$$

where e^- is a null one-form and

$$\omega_I = -i\delta_{\alpha\bar\beta}e^\alpha \wedge e^{\bar\beta} \ , \quad \omega_J = -e^1 \wedge e^2 - e^{\bar1} \wedge e^{\bar2} \ , \quad \omega_K = i(e^1 \wedge e^2 - e^{\bar1} \wedge e^{\bar2}) \ . \quad (4.12)$$

Clearly ω_I, ω_J and ω_K are Hermitian forms in the directions transverse to the light-cone. In what follows, we also set $\omega^1 = \omega_I$, $\omega^2 = \omega_J$ and $\omega^3 = \omega_K$.

The conditions that the gravitino KSE imposes on the spacetime geometry can be rewritten as

$$\hat\nabla_\mu e^- = 0 \ , \quad \hat\nabla_\mu(e^- \wedge \omega^{r'}) + 2\,\mathcal{C}^{s'}_\mu \epsilon^{r'}{}_{s't'}(e^- \wedge \omega^{t'}) = 0 \ . \quad (4.13)$$

The second equation can be thought as the Lorentzian analogue of the Quaternionic Kähler with torsion condition of [15]. The integrability conditions to these parallel transport equations are

$$\hat R_{\mu_1\mu_2,+\nu} = 0 \ , \quad -\hat R_{\mu_1\mu_2,}{}^k{}_i\omega^{r'}{}_{kj} + (j,i) + 2\mathcal{F}^{s'}_{\mu_1\mu_2}\epsilon^{r'}{}_{s't'}\omega^{t'}_{ij} = 0 \ . \quad (4.14)$$

In addition to this, the torsion H has to be anti-self-dual in 6 dimensions. The conditions for this can be written as

$$H_{+\alpha\beta} = H_{+\alpha}{}^\alpha = 0 \ , \quad H_{-+\bar\alpha} + H_{\bar\alpha\beta}{}^\beta = 0 \ , \quad H_{-1\bar1} - H_{-2\bar2} = 0 \ , \quad H_{-1\bar2} = 0 \ , \quad (4.15)$$

where $\epsilon_{-+1\bar12\bar2} = \epsilon_{013245} = -1$. Notice that from the 4-dimensional perspective, H_{+ij} is an anti-self-dual while H_{-ij} is a self-dual 2-form, respectively.

To specify the spacetime geometry, one has to solve (4.13) subject to (4.15). For this adapt a frame basis on the spacetime such that one of the light-cone frames is the parallel 1-form e^-, ie the metric is written as

$$ds^2 = 2e^-e^+ + \delta_{ij}e^ie^j \ . \quad (4.16)$$

The first condition in (4.13) implies that the dual vector field X to e^- is *Killing* and

$$de^- = i_X H \ . \quad (4.17)$$

From this, it is easy to see that the torsion 3-form can be written as

$$H = e^+ \wedge de^- + \frac{1}{2}H_{-ij}e^- \wedge e^i \wedge e^j + \tilde H \ , \quad \tilde H = \frac{1}{3!}\tilde H_{ijk}e^i \wedge e^j \wedge e^k \ . \quad (4.18)$$

Anti-self-duality of H relates the $\tilde H$ component to de^-. In particular, one has that

$$\tilde H = -\frac{1}{3!}(de^-)_{-\ell}\,\epsilon^\ell{}_{ijk}\,e^i \wedge e^j \wedge e^k \ . \quad (4.19)$$

This solves the first condition in (4.13). To solve the remaining three conditions, consider first the parallel transport equation in (4.13) along the light-cone directions. Since H_{+ij} is anti-self-dual, one has that

$$\mathcal{D}_+\omega^{r'} = \nabla_+\omega^{r'} + 2\,\mathcal{C}^{s'}_+\epsilon^{r'}{}_{s't'}\omega^{t'} = 0 \ . \quad (4.20)$$

This is a condition can be used to express $\mathcal{C}_+$ in terms of the geometry of spacetime. Next

$$\mathcal{D}_-\omega_{ij}^{r'} = \nabla_-\omega_{ij}^{r'} - H_-{}^k{}_{[i}\omega_{j]k}^{r'} + 2\mathcal{C}_-^{s'}\epsilon^{r'}{}_{s't'}\omega_{ij}^{t'} = 0 \ . \tag{4.21}$$

Since H_{-ij} is self-dual, this implies that it can be written as

$$H_{-ij} = w_{r'}\omega_{ij}^{r'} \ , \tag{4.22}$$

for some functions $w_{r'}$. Thus

$$\nabla_-\omega_{ij}^{r'} + w^{s'}\epsilon^{r'}{}_{s't'}\omega_{ij}^{t'} + 2\mathcal{C}_-^{s'}\epsilon^{r'}{}_{s't'}\omega_{ij}^{t'} = 0 \ . \tag{4.23}$$

This is interpreted as a condition which relates $\mathcal{C}_-^{s'}$ to the H_{-ij} components of the torsion. As a result, it can be solved to express H_{-ij} in terms of other fields and the geometry of spacetime.

To determine the conditions imposed on the geometry from the gravitino KSE in directions transverse to the lightcone, observe that a generic metric connection in 4 dimensions has holonomy contained in $Sp(1) \cdot Sp(1)$. Thus the only condition required is the identification of $Sp(1)$ part of the metric spacetime connection with the $Sp(1)$ part of induced connection from the Quaternionic Kähler manifold of the hyper-multiplets. This also follows from the integrability conditions (4.14).

Thus to summarize, the spacetime admits a null Killing vector field X whose rotation in the directions transverse to the light-cone is anti-self-dual. The geometry is restricted by (4.20). Furthermore, (4.23) relates the self-dual H_{-ij} component of the torsion to a component of the induced $Sp(1)$ connection from the Quaternionic Kähler manifold of the hypermultiplets. The metric and torsion of the spacetime can be written as

$$\begin{aligned}
ds^2 &= 2e^-e^+ + \delta_{ij}e^ie^j \ , \\
H &= e^+ \wedge de^- - \left(\frac{1}{16}\omega_{kl}^{r'}\nabla_-\omega^{s'kl}\epsilon_{r's'}{}^{t'} + \mathcal{C}_-^{t'}\right)\omega_{t'ij} \ e^- \wedge e^i \wedge e^j \\
&\quad - \frac{1}{3!}(de^-)_{-\ell}\ \epsilon^\ell{}_{ijk}\ e^i \wedge e^j \wedge e^k \ .
\end{aligned} \tag{4.24}$$

The remaining conditions that arise from the KSE are restrictions on the matter content of the theory. Let us begin with the gaugino KSE. To analyze the conditions, one can choose the gauge

$$A_+ = 0 \ . \tag{4.25}$$

In such a case, the components of the gauge connections do not depend on the coordinate adapted to the Killing vector field $X = \partial_u$. The components $F_{-i}^{a'}$ are not restricted by the KSE. In the directions transverse to the light-cone, the self-dual part of $F_{ij}^{a'}$ is given in terms of the moment maps while the anti-self-dual part is not restricted. So one can write

$$F^{a'} = F_{-i}^{a'}e^- \wedge e^i + \frac{1}{2}\mu_{r'}\omega^{r'} + (F^{\mathrm{asd}})^{a'} \ . \tag{4.26}$$

This is a Lorentzian version of the Hermitian-Einstein condition.

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Turning to the tensorini KSE, it is clear that in the gauge (4.25), the tensorini scalars are invariant under the isometries of the spacetime, ie they do not depend on the coordinate u. The 3-form field strengths are self-dual in 6 dimensions. This implies that

$$H^{\underline{M}}_{-\alpha\beta} = H^{\underline{M}}_{-\alpha}{}^{\alpha} = 0 \ , \quad H^{\underline{M}}_{-+\bar\alpha} - H^{\underline{M}}_{\bar\alpha\beta}{}^{\beta} = 0 \ , \quad H^{\underline{M}}_{+1\bar1} - H^{\underline{M}}_{+2\bar2} = 0 \ , \quad H^{\underline{M}}_{+1\bar2} = 0 \ . \quad (4.27)$$

Combining these conditions with those from the tensorini KSE, one finds that

$$H^{\underline{M}}_{+ij} = 0 \ . \tag{4.28}$$

$H^{\underline{M}}_{-ij}$ is *anti-self-dual* in the directions transverse to the light-cone and the remaining components are determined in terms of T. Therefore

$$H^{\underline{M}} = \frac{1}{2} H^{\underline{M}}_{-ij}\, e^- \wedge e^i \wedge e^j + T^{\underline{M}}_i e^- \wedge e^+ \wedge e^i - \frac{1}{3!} T^{\underline{M}}_{\ell} \epsilon^{\ell}{}_{ijk}\, e^i \wedge e^j \wedge e^k \ . \tag{4.29}$$

There are some further simplifications provided we use (2.6) to express the KSEs in terms of the fundamental fields. In particular, (4.6) implies that $\mathcal{C}^{r'}_+ = 0$ and so (4.20) leads to the geometric conditions

$$\nabla_+ \omega^{r'} = 0 \ , \quad r' = 1, 2, 3 \ . \tag{4.30}$$

In addition, , $T^{\underline{M}}_i = x^{\underline{M}}_{\underline{r}} \partial_i v^r$. Substituting this in (4.29) most of the components of $H^{\underline{M}}$ are determined in terms of the scalars. Furthermore, the conditions of the hyperini KSE in the gauge (4.25) imply that the scalars of the multiplet are invariant under the action of isometries generated by X, ie

$$D_+ \phi^I = \partial_u \phi^I = 0 \ . \tag{4.31}$$

The remaining restrictions give a holomorphicity-like condition for the imbedding scalars.

5   N=2 non-compact

There are two cases with $N = 2$ supersymmetry distinguished by the isotropy group of the Killing spinors. If the isotropy group is non-compact $U(1) \cdot SU(2) \ltimes \mathbb{H}$, the two Killing spinors are

$$\epsilon_1 = 1 + e_{1234} \ , \quad \epsilon_2 = i(1 - e_{1234}) = \rho^1 \epsilon_1 \ . \tag{5.1}$$

Therefore, the additional conditions on the fields which arise from the second Killing spinor can be expressed as the requirement that the KSE must commute with the Clifford algebra operation ρ^1.

5.1   Gravitino

It is clear that the gravitino KSE commutes with ρ^1, iff

$$\mathcal{C}^2 = \mathcal{C}^3 = 0 \ . \tag{5.2}$$

Equivalently, the gravitino KSE implies that the holonomy of the supercovariant connection is included in $U(1) \cdot Sp(1) \ltimes \mathbb{H}$, $\mathrm{hol}(\mathcal{D}) \subseteq U(1) \cdot Sp(1) \ltimes \mathbb{H}$. The restrictions that this imposes on the geometry will be investigated later.

5.2 Gaugini

The gaugini KSE commutes with ρ^1, iff

$$\mu_2 = \mu_3 = 0 \ . \tag{5.3}$$

These restrictions are in addition to the conditions given in (4.2).

5.3 Tensorini

A direct substitution of the second Killing spinor into the tensorini KSE reveals that there are no additional conditions to those given in (4.3). As we have mentioned the tensorini KSE commutes with all ρ Clifford algebra operations.

5.4 Hyperini

Combining the restrictions imposed by the second Killing spinor with those presented in (4.5) for the first Killing spinor, one finds

$$V_{+}^{aA} = 0 \ , \quad V_{\alpha}^{a1} = 0 \ , \quad V_{\bar\alpha}^{a2} = 0 \ . \tag{5.4}$$

5.5 Geometry

The form spinor bi-linears are given in (4.13). The only different is that now the full content of gravitino KSE can be expressed as

$$\hat\nabla e^- = 0 \ , \quad \hat\nabla(e^- \wedge \omega) = 0 \ ,$$
$$\hat\nabla(e^- \wedge \omega^2) - 2\,\mathcal{C}e^- \wedge \omega^3 = 0 \ , \hat\nabla(e^- \wedge \omega^3) + 2\,\mathcal{C}e^- \wedge \omega^2 = 0 \ , \tag{5.5}$$

where we have set $\omega = \omega^1$ and $\mathcal{C} = \mathcal{C}^1$, ie the form $e^- \wedge \omega$ is covariantly constant with respect to the connection with skew-symmetric torsion only.

It is clear that the spacetime admits a null Killing vector field X, the dual of the 1-form e^-, and that (4.17) is valid. The metric and torsion 3-form can be written as in (4.16) and (4.18), respectively.

To continue, let us investigate the remaining 3 parallel transport equations in (5.5). As in the previous $N = 1$ case, the parallel transport equations along the $+$ light-cone direction leads to (4.20) but with $\mathcal{C}^2 = \mathcal{C}^3 = 0$. Thus, one has

$$\nabla_+\omega_{ij}^1 = 0 \ , \quad \nabla_+\omega_{ij}^2 - 2\,\mathcal{C}_+\omega_{ij}^3 = 0 \ , \quad \nabla_+\omega_{ij}^3 + 2\,\mathcal{C}_+\omega_{ij}^2 = 0 \ . \tag{5.6}$$

The first condition is a restriction on the geometry. The second can be solved for $\mathcal{C}_+$ to give

$$\mathcal{C}_+ = \frac{1}{8}(\omega^3)^{ij}\nabla_+\omega_{ij}^2 \ . \tag{5.7}$$

The third equation in (5.6) is automatically satisfied. Using that H_{-ij} is self-dual and

$$\hat\nabla_-\omega_{ij} = \nabla_-\omega_{ij} - H_-{}^k{}_{[i}\omega_{j]k} = 0, \tag{5.8}$$

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one can solve for H_{-ij} to find

$$H_{-ij} = -\nabla_- \omega_{ik} I^k{}_j \ . \tag{5.9}$$

Two remaining conditions along the $-$ light-cone direction can be used to express $\mathcal{C}_-$ in terms of the geometry and give some additional restrictions on the geometry of spacetime. In particular, one has

$$\mathcal{C}_- = \frac{1}{8} \nabla_- \omega_{ij}^2 \omega^{3ij} \ ,$$
$$\nabla_- \omega_{ij}^2 - \nabla_- \omega_{k[i}^1 (I^3)^k{}_{j]} - \frac{1}{4} \nabla_- \omega_{k\ell}^2 \omega^{3k\ell} \omega_{ij}^3 = 0 \ ,$$
$$\nabla_- \omega_{ij}^3 + \nabla_- \omega_{k[i}^1 (I^2)^k{}_{j]} + \frac{1}{4} \nabla_- \omega_{k\ell}^2 \omega^{3k\ell} \omega_{ij}^2 = 0 \ . \tag{5.10}$$

The conditions transverse to the light-cone give

$$\tilde{H} = -i_I \tilde{d}\omega \ , \tag{5.11}$$

where $\tilde{d}$ is the exterior derivative projected in directions transverse to the light-cone. This together with the anti-self-duality condition for H turn (4.19) into a condition on the geometry of spacetime

$$(de^-)_{-\ell} \, \epsilon^\ell{}_{ijk} = (i_I \tilde{d}\omega)_{ijk} \ . \tag{5.12}$$

The other two parallel transport equations are automatically satisfied provided that the $U(1)$ part of the curvature tensor of the spacetime connection with torsion is identified with the curvature of $U(1)$ connection $\mathcal{C}$. To see this observe that the integrability conditions of the gravitino KSE can be written as

$$\hat{R}_{\mu_1\mu_2,+\nu} = 0 \ , \quad \hat{R}_{\mu_1\mu_2,ki} I^k{}_j - \hat{R}_{\mu\nu,kj} I^k{}_i = 0 \ ,$$
$$-\hat{R}_{\mu_1\mu_2,ki} J^k{}_j + \hat{R}_{\mu_1\mu_2,kj} J^k{}_i - 2\mathcal{F}_{\mu_1\mu_2} \omega_{ij}^3 = 0 \ . \tag{5.13}$$

The second condition implies that the holonomy of the $\hat{\nabla}$ connection in the directions transverse to the ligh-cone is contained in $U(2) = U(1) \cdot Sp(1)$. The last condition identifies the $U(1)$ part of the curvature with the curvature of $\mathcal{C}$.

To *summarize*, the gravitino KSE implies that the metric and torsion can be written as

$$ds^2 = 2e^- e^+ + \delta_{ij} e^i e^j \ ,$$
$$H = e^+ \wedge de^- - \nabla_- \omega_{ik} I^k{}_j \, e^- \wedge e^i \wedge e^j - \frac{1}{3!} (de^-)_{-\ell} \, \epsilon^\ell{}_{ijk} \, e^i \wedge e^j \wedge e^k \ . \tag{5.14}$$

Of course as in the $N = 1$ case, the spacetime admits a null Killing vector field X which also determines components of H and the geometric condition (4.20) is satisfied. Furthermore, one has to impose the geometric conditions (5.10), (5.12) and the restrictions implied by (5.13).

As we have mentioned the tensorini KSE does not impose any new conditions on the matter field. As a result, the restrictions are summarized in (4.3) and the fields are expressed as in (4.29).

The gaugino KSE gives (5.3). So in the gauge $A_+ = 0$, one has

$$F^{a'} = F^{a'}_{-i} \, e^- \wedge e^i + \frac{1}{2} \mu \, \omega + (F^{\text{asd}})^{a'} \, , \quad \mu^2 = \mu^3 = 0 \, , \tag{5.15}$$

where $\mu = \mu^1$.

The hypernini KSE imposes a restriction on the $+$ lightcone direction. The rest of the conditions are Cauchy-Riemann type of equations on the scalars.

As in the $N = 1$ case, expressing the KSEs in terms of the fundamental fields (2.6), one can improve somewhat on the solutions to the KSEs. In particular, the hyperini KSE condition $D_+\phi = 0$, (4.6), implies that $\mathcal{C}_+ = 0$. Using (5.6) gives rise to the geometric conditions

$$\nabla_+ \omega^1_{ij} = \nabla_+ \omega^2_{ij} = \nabla_+ \omega^3_{ij} = 0 \, . \tag{5.16}$$

Writing $X = \partial_u$ and taking the gauge $A_+ = 0$, one again concludes that ϕ are independent from u, (4.31).

6 N=2 compact

6.1 Gravitino

The 2 Killing spinors with isotropy group $Sp(1)$, table 1, can be chosen as

$$\epsilon_1 = 1 + e_{1234} \, , \quad \epsilon_2 = e_{15} + e_{2345} \, . \tag{6.1}$$

The full content of the gravitino KSE is

$$\text{hol}(\mathcal{D}) \subseteq Sp(1) \, . \tag{6.2}$$

The implications that this condition has on the spacetime geometry will be investigated later.

6.2 Gaugini

Evaluating the gaugini KSE on $e_{15} + e_{2345}$, one finds

$$-2F_{1\bar{2}} + \mu^2 + i\mu^3 = 0 \, , \quad -F_{1\bar{1}} + F_{2\bar{2}} + i\mu^1 = 0 \, , \quad F_{-i} = 0 \, . \tag{6.3}$$

Combining the above conditions with those in (4.2), we get that

$$F^{a'}_{ab} = 0 \, , \quad F^{a'}_{ai} = 0 \, , \quad F^{a'}_{ij} = -\epsilon_{ijk}\mu^{a'k} \, , \quad a = -, +, \tilde{1} \, , \tag{6.4}$$

where $\epsilon_{245} = -1$. Each of the indices a and i labels 3 real directions, $i = 4, \tilde{2}, 5$, where we have used $\tilde{1}$ and $\tilde{2}$ to distinguish the real directions from the complex directions 1 and 2 which naturally appear in the various conditions which arise from the KSEs. In addition, the $r' = 1, 2, 3$ index of μ has been replaced with $k = 4, 2, 5$ after an appropriate adjustment of the ranges and identification of the components of μ.

6.3 Tensorini

A direct substitution of $e_{15} + e_{2345}$ in the tensorini KSE gives

$$T_{-}^{\underline{M}} = 0 , \quad H_{-1\bar{1}}^{\underline{M}} - H_{-2\bar{2}}^{\underline{M}} = 0 , \quad H_{-1\bar{2}}^{\underline{M}} = 0 ,$$
$$T_{\bar{\alpha}}^{\underline{M}} + \frac{1}{2} H_{-+\bar{\alpha}} + \frac{1}{2} H_{\bar{\alpha}\beta}{}^{\beta} = 0 . \tag{6.5}$$

Combining these conditions with those derived for $1 + e_{1234}$ and using the self-duality of $H^{\underline{M}}$, one finds that

$$T_{\mu}^{\underline{M}} = 0 , \quad H_{\mu\nu\rho}^{\underline{M}} = 0 . \tag{6.6}$$

So the tensorini KSE vanishes identically. As a result all 8 supersymmetries are preserved. In turn using the expression of T and H in terms of the physical fields (2.6), one finds that the scalars are constant and 3-form field strengths of the tensor multiplet vanish.

6.4 Hyperini

Evaluating the hyperini KSE on $e_{15} + e_{2345}$, one finds that

$$V_{-}^{aA} = 0 , \quad -V_{2}^{a1} + V_{1}^{a2} = 0 , \quad V_{\bar{1}}^{a1} + V_{\bar{2}}^{a2} = 0 . \tag{6.7}$$

Combining these conditions with those in (4.5), we get

$$V_{a}^{aA} = 0 , \quad a = -, +, \tilde{1} . \tag{6.8}$$

The remaining conditions can be derived by substituting (6.8) in either (4.5) or (6.7).

Expressing the KSE in terms of the physical fields as in (2.6), one finds that (6.8) implies

$$D_a \phi^I = 0 , \quad a = -, +, \tilde{1} . \tag{6.9}$$

The hypermultiplet scalars do not depend on 3 spacetime directions.

6.5 Geometry

The algebraic independent form bi-linears are

$$e^a , \quad a = -, +, \tilde{1} ; \quad e^i , \quad i = 4, \tilde{2}, 5 , \tag{6.10}$$

where e^a and e^i are 1-forms. The conditions implied by the gravitino Killing spinor equation can be rewritten as

$$\hat{\nabla}_\mu e^a = 0 ,$$
$$\hat{\nabla}_\mu e^i + 2\epsilon^i{}_{jk} \mathcal{C}_\mu^j e^k = 0 , \tag{6.11}$$

where as in the gaugini case the indices r', s' and t' have been replaced with i, j and k, the ranges have been adjusted, and the components of $\mathcal{C}$ have been appropriately identified.

It is clear that the spacetime admits a $3 + 3$ "split". In particular, the tangent space, TM, of spacetime decomposes as

$$TM = I + \xi \; , \tag{6.12}$$

where I is a topologically trivial vector bundle spanned by the vector fields associated to the three 1-forms e^a.

The 1-forms e^a and e^i can be used as a spacetime frame and write the metric as

$$ds^2 = \eta_{ab}e^a e^b + \delta_{ij}e^i e^j \; . \tag{6.13}$$

Let us first focus on the first equation in (6.11). This implies that the associated vector fields to e^a are *Killing*. In addition using the anti-self-duality of H, all the components of H can be determined in terms of e^a and its first derivatives. In particular, one has

$$de^a = \eta^{ab}i_b H \; , \tag{6.14}$$

where $\eta^{ab} = g(e^a, e^b)$, and so

$$H_{a_1 a_2 a_3} = \eta_{a_1 b}de^b_{a_2 a_3} \; , \quad H_{a_1 a_2 i} = \eta_{a_1 b}de^b_{a_2 i} \; , \quad H_{aij} = \eta_{ab}de^b_{ij} \; . \tag{6.15}$$

The first two equations relate the components of H to the commutators of two Killing vector fields projected along the e^a and e^i directions, respectively, see [12, 13]. The anti-self-duality condition for H gives

$$H_{a_1 a_2 a_3}\epsilon^{a_1 a_2 a_3} = H_{ijk}\epsilon^{ijk} \; , \quad \epsilon_b{}^{a_1 a_2}H_{a_1 a_2 i} = -\epsilon_i{}^{jk}H_{bjk} \; , \tag{6.16}$$

where $\epsilon_{013} = \epsilon_{245} = 1$. Thus H can be rewritten as

$$H = K - \star K \; , \quad K = \frac{1}{3!}H_{a_1 a_2 a_3}e^{a_1} \wedge e^{a_2} \wedge e^{a_3} + \frac{1}{2}H_{ia_1 a_2}e^i \wedge e^{a_1} \wedge e^{a_3} \; , \tag{6.17}$$

subject to the geometric condition

$$(de_{a_1})_{a_2 i_1}\epsilon^{a_1 a_2}{}_{a_3} = -\epsilon_{i_1}{}^{i_2 i_3}(de_{a_3})_{i_2 i_3} \; . \tag{6.18}$$

Returning to the second equation in (6.11), one finds that it is equivalent to

$$\begin{aligned}
\nabla_b e^i_j - \frac{1}{2}H^i{}_{bj} + 2\epsilon^i{}_{kj}\mathcal{C}^k_b &= 0 \; , \\
\nabla_j e^i_k - \frac{1}{2}H^i{}_{jk} + 2\epsilon^i{}_{sk}\mathcal{C}^s_j &= 0 \; .
\end{aligned} \tag{6.19}$$

The first condition again express a component of H in terms of the geometry and $\mathcal{C}$. Substituting the expression we have for H in (6.15), one finds

$$\nabla_a e^i_j + 2\epsilon^i{}_{jk}\mathcal{C}^j_a e^k = -\frac{1}{2}\eta_{ab}\, de^b_{kj}\, \delta^{ki} \; . \tag{6.20}$$

The last condition in (6.19) identifies the spin connection $\hat{\Omega}$ of the spacetime in directions transverse to the Killing with the induced $Sp(1)$ connection of the scalars. This can also

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be seen by looking at the integrability conditions of the gravitino KSE. In particular, one has

$$\hat{R}_{AB,aC} = 0 \ , \quad \hat{R}_{AB,j_1 j_2} = -2\mathcal{F}_{AB}^{k} \epsilon_{k j_1 j_2} \ . \tag{6.21}$$

These two conditions follow from the integrability conditions of (6.11) on e^a and e^i, respectively.

Moreover, expressing the KSEs in terms of the physical fields and using the restrictions imposed by the gaugini and hyperini KSEs, one also finds

$$\hat{R}_{aB,CD} = 0 \ . \tag{6.22}$$

Similarly, one also has that

$$\mathcal{C}_a^i = 0 \ , \tag{6.23}$$

and so (6.20) turns into a condition on the geometry. It is clear that the only non-trivial components of the curvature with torsion are those along the transverse to the Killing vector directions and all of them are specified in terms of the curvature of $\mathcal{C}$.

To summarize, the spacetime admits 3 Killing vector fields and the torsion H is completely determine in terms of these and their first derivatives. In particular, one has

$$ds^2 = \eta_{ab}e^a e^b + \delta_{ij}e^i e^j \ ,$$
$$H = K - \star K \ , \quad K = \frac{1}{3!}H_{a_1 a_2 a_3}e^{a_1} \wedge e^{a_2} \wedge e^{a_3} + \frac{1}{2}H_{i a_1 a_2}e^i \wedge e^{a_1} \wedge e^{a_3} \ . \tag{6.24}$$

In addition, the spacetime geometry is restricted by (6.18), (6.20) and the last condition in (6.19) or equivalently (6.21). The conditions imposed by the remaining 3 KSEs are self-explanatory.

6.5.1 An example

Under some additional assumptions, the geometry of spacetime can be described in terms of principal bundles. In particular, one can take either that H is closed, $dH = 0$, or that the algebra of vector fields associated with e^a closes under Lie brackets. These two assumptions are related. Following the results of [14], if H is closed and the commutator of the vector fields does not close under Lie brackets, then the spacetime admits at least an additional parallel vector field. In turn, the holonomy of the supercovariant connection reduces to subgroup of $U(1)$. Such solutions admit at least 4 parallel spinors and they are investigated later. So if one insists on solutions with strictly 2 parallel spinors, $dH = 0$ implies that the algebra of the three isometries closes under Lie brackets. So suppose that the algebra of the 3 Killing vector fields closes. In analogy with the results of [12], the spacetime can be thought as a principal bundle with fibre group which has Lie algebra

$$\mathbb{R}^{2,1} \ , \quad \mathfrak{sl}(2,\mathbb{R}) \ , \tag{6.25}$$

where we have used the classification of Lorentzian Lie algebras [19, 20]. The closure property of the Lie algebra of the 3 Killing vector fields requires that

$$H_{abi} = 0 \ . \tag{6.26}$$

In turn, the anti-self-duality of H requires that

$$H_{aij} = 0 \ . \tag{6.27}$$

In [12] this component of H was identified with the curvature of the principal bundle. Thus if $H_{aij} = 0$, the spacetime is locally a product $G \times \Sigma$, where G is either $\mathbb{R}^{2,1}$ or $SL(2, \mathbb{R})$ and Σ is a 3-dimensional Riemannian manifold. The curvature of Σ is related to the curvature of $\mathcal{C}$ as in (6.21). Such a condition is not trivial as it requires the existence of a metric on Σ whose curvature is equal to a prescribed quantity. A related example is the Calabi conjecture. However there are solutions. For example, $SL(2, \mathbb{R}) \times S^3$ is a solution with the radii of the two factors equal, the scalars constant, and with vanishing gauge connection.

7 N=4 non-compact

The Killing spinors are the $Sp(1) \ltimes \mathbb{H}$-invariant spinors of table 1. These can be rewritten

$$1 + e_{1234} \ , \quad \rho^1 (1 + e_{1234}) \ , \quad \rho^2 (1 + e_{1234}) \ , \quad \rho^3 (1 + e_{1234}) \ . \tag{7.1}$$

Therefore the KSEs commute with the Clifford algebra operations $\rho^{r'}$. We shall use this together with the conditions imposed on backgrounds preserving 1 supersymmetry to derive all the conditions implied by the KSEs in this case.

7.1 Gravitino

The gravitino KSE commutes with the $\rho^{r'}$ operations iff

$$\mathcal{C} = 0 \ . \tag{7.2}$$

As a result the curvature $\mathcal{F}$ of $\mathcal{C}$ vanishes. Thus the full content of the gravitino KSE can be expressed as $\mathrm{hol}(\hat{\nabla}) \subseteq Sp(1) \ltimes \mathbb{H}$. The restrictions that this condition imposes on the spacetime geometry will be examined later.

7.2 Gaugini

The KSE commute with $\rho^{r'}$, iff

$$\mu_1 = \mu_2 = \mu_3 = 0 \ . \tag{7.3}$$

These are in addition to the conditions given in (4.2). Thus, we have that

$$F^{a'} = F^{a'}_{-i} e^- \wedge e^i + (F^{\mathrm{asd}})^{a'} \ . \tag{7.4}$$

7.3 Tensorini

The tensorini KSE commutes with the Clifford algebra operations $\rho^{r'}$. Thus there are no additional conditions to those given in (4.3)

7.4 Hyperini

In addition to the conditions (5.4), one finds

$$V_{\bar{\alpha}}^{\underline{a1}} = 0 \ , \quad V_{\alpha}^{\underline{a2}} = 0 \ . \tag{7.5}$$

Thus the only non-vanishing component is

$$V_{-}^{\underline{aA}} \tag{7.6}$$

Imposing the conditions of the hyperini KSE on the physical fields using (2.6), one finds that the only non-vanishing derivative on the scalars is

$$D_{-}\phi^{\underline{I}} \ . \tag{7.7}$$

Thus the scalars depend only on one light-cone direction.

7.5 Geometry

The spinor bi-linears are the same as those of the $N = 2$ non-compact case. The important difference here is that $\mathcal{C} = 0$ and so the conditions imposed by gravitino KSE can be rewritten as

$$\hat{\nabla}e^- = 0 \ , \quad \hat{\nabla}(e^- \wedge \omega^{r'}) = 0 \ . \tag{7.8}$$

The solution to these conditions is similar to that of the non-compact $N = 2$. So one writes

$$\begin{aligned}
ds^2 &= 2e^- e^+ + \delta_{ij}e^i e^j \ , \\
H &= e^+ \wedge de^- - \frac{1}{16}\omega_{kl}^{r'}\nabla_-\omega^{s'kl}\epsilon_{r's'}{}^{t'}\,\omega_{t'ij}\ e^- \wedge e^i \wedge e^j \\
&\quad - \frac{1}{3!}(de^-)_{-\ell}\,\epsilon^{\ell}{}_{ijk}\ e^i \wedge e^j \wedge e^k \ .
\end{aligned} \tag{7.9}$$

We have used the anti-self-duality of H to relate the $\tilde{H}$ component to de^- as in (4.19).

It remains to find the geometric conditions on the spacetime. We have already dealt with the first condition in (7.8). To solve the last 3 conditions in (7.8), one has that

$$\hat{\nabla}_+\omega^{r'} = \nabla_+\omega^{r'} = 0 \ . \tag{7.10}$$

This is a condition on the geometry. Furthermore, one has that

$$\nabla_-\omega_{ij}^{r'} - H_-{}^k{}_{[i}\omega_{j]k}^{r'} = 0 \ . \tag{7.11}$$

This together with the self-duality of H_{-ij} can be used to express H_{-ij} in terms of the geometry as in (7.9). There are no conditions on the geometry along this light-cone direction.

Next, the conditions along the transverse to light-cone directions give

$$\tilde{H} = -i_{I^{r'}}\tilde{d}\omega^{r'} \ , \quad \text{(no r' summation) .} \tag{7.12}$$

Although these may appear as three independent conditions actually they are not. One of them implies the other two. In turn, this condition together with (4.19) imply

$$de_{-j}^-\,e^j{}_{i_1 i_2 i_3} = (i_{I^{r'}}\tilde{d}\omega^{r'})_{i_1 i_2 i_3} \ , \quad \text{(no r' summation) .} \tag{7.13}$$

This is another condition on the geometry. The restrictions on the fields imposed by the other 3 KSEs have already been explained.

7.6 N=3 descendant

Unlike all other cases, the $N = 4$ backgrounds with $Sp(1) \ltimes \mathbb{H}$-invariant parallel spinors exhibit an independent descendant with 3 supersymmetries. We have already argued that the conditions on the fields implied by gravitino, gaugini and tensorini KSEs remain the same as those for backgrounds with 4 Killing spinors (7.1). Different conditions appear only in the analysis of hyperini KSE.

The 3 Killing spinors have been given in (3.7). A direct substitution into the hyperini KSE reveals that

$$V_{+}^{aA} = 0 , \quad V_{\alpha}^{a1} = V_{\bar{\alpha}}^{a2} = 0 , \quad V_{\bar{1}}^{a1} - V_{2}^{a2} = 0 , \quad V_{\bar{2}}^{a1} + V_{1}^{a2} = 0 . \quad (7.14)$$

These conditions are different from those we have found in (5.4) and (7.5) which arise for the case of 4 supersymmetries. It is straightforward to express the above conditions in terms of the physical fields using (2.6). For example, it is easy to see that the first condition implies (4.31). The analysis for the geometry of the spacetime we have made in the previous section remains unaltered. Of course the scalars of the hyperini KSE satisfy different conditions from those of backgrounds with 4 supersymmetries.

8 N=4 compact

The Killing spinors are the $U(1)$-invariant spinors of table 1. These can be rewritten as

$$1 + e_{1234} , \quad e_{15} + e_{2345} , \quad \rho^1(1 + e_{1234}) , \quad \rho^1(e_{15} + e_{2345}) . \quad (8.1)$$

Thus the conditions on the fields that arise from the KSEs are those we have found for the $Sp(1)$-invariant Killing spinors, and those required for the KSEs to commute with the Clifford algebra operation ρ^1.

8.1 Gravitino

The Clifford algebra operation ρ^1 commutes with the gravitino KSE provided that

$$\mathcal{C}^2 = \mathcal{C}^3 = 0 . \quad (8.2)$$

As in previous cases, the full content of the gravitino KSE can be expressed as $\mathrm{hol}(\mathcal{D} \subseteq U(1)$. The geometry of spacetime will be examined below.

8.2 Gaugini

The gaugini KSE commutes with ρ^1 iff $\mu^2 = \mu^3 = 0$. Combining this with (6.4), one finds

$$F_{2\bar{2}}^{a'} + i\mu^{a'} = 0 , \quad (8.3)$$

where after suppressing the gauge index $\mu = \mu^1$.

8.3 Tensorini

The tensorini KSE commutes with all the Clifford algebra $\rho^{r'}$ operators. Since both $1 + e_{1234}$ and $e_{15} + e_{2345}$ are Killng spinors, one concludes that all 8 supersymmetries are preserved. Thus $T^{\underline{M}} = H^{\underline{M}} = 0$ as in (6.6). In turn, the tensorini multiplet scalars are constant and the 3-form field strengths vanish.

8.4 Hyperini

To find the conditions that arise from the hypernini KSE, one has to simultaneously impose (6.7) and (5.4). Thus one has that

$$V^{\underline{aA}}_a = 0 \ , \quad a = -, +, 1, \bar{1} \ , \tag{8.4}$$

and

$$V^{\underline{a1}}_2 = V^{\underline{a2}}_{\bar{2}} = 0 \ . \tag{8.5}$$

The only non-vanishing components are $V^{\underline{a1}}_{\bar{2}}$ and $V^{\underline{a2}}_2$.

Using (2.6), the above conditions can be expressed in terms of the physical fields as

$$D_a \phi^{\underline{I}} = 0 \ , \quad a = -, +, 1, \bar{1} \ , \tag{8.6}$$

and

$$D_2 \phi^{\underline{I}} E^{\underline{a1}}_{\underline{I}} = D_{\bar{2}} \phi^{\underline{I}} E^{\underline{a2}}_{\underline{I}} = 0 \ , \tag{8.7}$$

respectively. Clearly, the scalar fields do not depend on 4 spacetime directions. The last two conditions are Cauchy-Riemann type of equations along the remaining two directions.

8.5 Geometry

A basis for algebraically independent bi-linears is spanned by the 1-forms

$$e^a \ , \quad a = -, +, 1, \bar{1} \ , \quad e^i \ , \quad i = 2, \bar{2} \ . \tag{8.8}$$

The gravitino KSE can be rewritten as

$$\hat{\nabla} e^a = 0 \ , \quad \hat{\nabla} e^i - 2\mathcal{C} \, \epsilon^i{}_j e^j = 0 \ , \tag{8.9}$$

where we have set $\mathcal{C} = \mathcal{C}^1$.

As in previous cases, the first equation again implies that the vector fields X_a associated with the 1-forms e^a are Killing and

$$i_a H = \eta_{ab} de^b \ . \tag{8.10}$$

It is clear that the spacetime admits a $4 + 2$ split. In particular, the tangent space $TM = I \oplus \xi$, where now I is a rank 4 trivial vector bundle spanned by the 4 Killing vectors X_a.

The second equation in (8.9) is equivalent to requiring that

$$(\nabla_a e^i)_j - \frac{1}{2} H^i{}_{aj} - 2\mathcal{C}_a \epsilon^i{}_j = 0 ,$$
$$(\nabla_j e^i)_k - 2\mathcal{C}_j \epsilon^i{}_k = 0 . \tag{8.11}$$

In turn, the first condition in (8.11) gives

$$(\nabla_a e^i)_j - 2\mathcal{C}_a \epsilon^i{}_j = -\frac{1}{2}\eta_{ab}(de^b)_{kj}\delta^{ki} , \tag{8.12}$$

as some components H are determined in terms of $\mathcal{C}$, and both the e^a and e^i bi-linears and their first derivatives. In addition, H is anti-self dual. This in turn implies that

$$H_{aij} = \frac{1}{3!}\epsilon_{ij}\,\epsilon_a{}^{b_1 b_2 b_3} H_{b_1 b_2 b_3} , \quad H_{a_1 a_2 i} = \frac{1}{2}\epsilon_{a_1 a_2}{}^{b_1 b_2}\epsilon_i{}^j H_{b_1 b_2 j} , \tag{8.13}$$

where $\epsilon_{2\bar{2}} = i$ and $\epsilon_{-+1\bar{1}} = i$. As all components of H are determined in terms of e^a and its first derivative, this leads to more restrictions on the geometry of spacetime. These can be expressed as

$$de^a_{ij} = \frac{1}{3!}\epsilon_{ij}\,\epsilon^{ab_1 b_2}{}_{b_3} de^{b_3}_{b_1 b_2} , \quad de^{a_1}_{a_2 i} = \frac{1}{2}\epsilon_{a_2 b_2}{}^{a_1 b_1}\epsilon_i{}^j de^{b_2}_{b_1 j} . \tag{8.14}$$

Observe that the rhs of the first equation depends on the structure constants of the algebra of the 4 Killing vector fields.

The last condition in (8.11) identifies the spacetime connection along the directions transverse to the Killing with a $U(1)$ component of the induced $Sp(1)$ quaternionic Kähler connection. This can also be seen by investigating the integrability conditions of (8.9). In particular, one finds that

$$\hat{R}_{AB,aC} = 0 , \quad \hat{R}_{\mu\nu,j_1 j_2} = -2\mathcal{F}_{\mu\nu}\epsilon_{j_1 j_2} . \tag{8.15}$$

The derivation of these conditions is similar to that of the $Sp(1)$ holonomy case.

There are some additional simplifications provided we use (2.6) to express the above conditions in terms of the physical fields. In particular using the hypernini and gaugini KSEs, one finds that apart from (8.15)

$$\hat{R}_{aB,CD} = 0 . \tag{8.16}$$

Similarly $\mathcal{C}_a = 0$ and so (8.12) becomes a condition on the geometry of spacetime.

8.5.1   Fibration

The KSEs do not imply that the algebra of 4 Killing vector field closes. Nevertheless, a large class of examples can be constructed by imposing closure of this algebra. As it has been explained in [14] and further discussed in the compact $N = 2$ case, if $dH = 0$ and one insists in the existence of strictly 4 parallel spinors, then the algebra of 4 Killing

vector fields closes. So the closure of the algebra is a natural assumption to make specially in the absence of gauge fields. In turn, the closure of the algebra implies

$$H_{abi} = 0 \ . \tag{8.17}$$

The Lie algebra of the Killing vector fields must be isomorphic [19, 20] to one of the following

$$\mathbb{R}^{3,1} \ , \quad \mathfrak{sl}(2,\mathbb{R}) \oplus \mathfrak{u}(1) \ , \quad \mathbb{R} \oplus \mathfrak{su}(2) \ , \quad \mathfrak{cw}_4 \ . \tag{8.18}$$

The spacetime can be interpreted as a principal bundle with fibre group, which has Lie algebra one of those in (8.18), and base space a 2-dimensional manifold B. Moreover it admits a principal bundle connection $\lambda^a = e^a$ with curvature given by de_{ij}^a. Unlike the $N = 2$ case, if the fibre group is not abelian, the fibre twists over B because of the first equation in (8.14). In the abelian case, the spacetime is locally a product $\mathbb{R}^{3,1} \times B$. Finally the Riemann curvature of B must be identified with the curvature of the $U(1)$ connection $\mathcal{C}$.

9 Trivial isotropy group

Backgrounds with parallel spinors which have a trivial isotropy group admit 8 parallel spinors. The spacetime is a Lorentzian Lie group with anti-self-dual structure constants. These have been classified in a similar context in [5]. In particular, the spacetime is locally isometric to

$$\mathbb{R}^{5,1} \ , \quad AdS_3 \times S^3 \ , \quad CW_6 \ , \tag{9.1}$$

where the radii of AdS_3 and S^3 are equal, and the structure constants of CW_6 are given by a constant self-dual 2-form on $\mathbb{R}^4$. Moreover

$$\mathcal{F}(\mathcal{C}) = 0 \ . \tag{9.2}$$

This concludes the conditions which arise from the gravitino KSE.

The gaugino KSE implies that the gauge field strength vanishes and $\mu^{r'} = 0$. The tensorini implies that the 3-form field strengths vanish and the scalars are constants. Similar hyperini KSE implies that the scalars are constant. In turn using (2.6), the latter gives $\mathcal{C} = 0$.

9.1 Descendants

The case of trivial isotropy group has descendants. In particular, the KSEs allow for backgrounds with 1,2,3 and 4 supersymmetries. However none of them is independent from the backgrounds and their descendants we have examined in previous cases. The proof of this is required to establish the results outlined in section 3.4. Here we shall not describe all the steps of the proof. Instead, we shall focus on one case. The rest follow in a similar way. In particular, let us consider the descendants with 3 supersymmetries for

which the Killing spinors are given in (3.10). To establish that there are no independent descendants, we have to solve the hyperini KSE for the spinors given in (3.10). The first two spinors give

$$
\begin{aligned}
V_+^{aA} = 0 \ , \quad & -V_1^{a1} + V_{\bar{2}}^{a2} = 0 \ , \quad V_2^{a1} + V_{\bar{1}}^{a2} = 0 \ , \\
V_-^{aA} = 0 \ , \quad & -V_2^{a1} + V_{\bar{1}}^{a2} = 0 \ , \quad V_{\bar{1}}^{a1} + V_{\bar{2}}^{a2} = 0 \ ,
\end{aligned}
\tag{9.3}
$$

which follows from (4.5) and (6.7). Evaluating the hyperini KSE on the third spinor in (3.10), one finds

$$
\begin{aligned}
& c_1 V_1^{a1} + c_1 V_{\bar{2}}^{a2} = 0 \ , \quad -c_1 V_2^{a1} + c_1 V_{\bar{1}}^{a2} = 0 \ , \\
& ic_2 V_{\bar{1}}^{a1} - c_3 V_{\bar{2}}^{a1} - ic_2 V_{\bar{2}}^{a2} + c_3 V_{\bar{1}}^{a2} = 0 \ , \\
& ic_2 V_2^{a1} + c_3 V_1^{a1} + ic_2 V_1^{a2} + c_3 V_2^{a2} = 0 \ .
\end{aligned}
\tag{9.4}
$$

It is clear that if $c_1 \neq 0$, then the V's vanish and so the hyperini KSE preserves all supersymmetry. On the other hand if $c_1 = 0$, it has been argued in section 3.4 that one can always set $c_3 = 0$. Setting $c_3 = 0$ in the last two conditions in (9.4), one finds that

$$
V_{\bar{1}}^{a1} - V_{\bar{2}}^{a2} = 0 \ , \quad V_2^{a1} + V_1^{a2} = 0 \ .
\tag{9.5}
$$

Comparing this with (9.3), we again find that all V's vanish. Thus again the hyperini KSE preserves all supersymmetry and so there is not a new descendant.

10   Conclusions

We have solved the KSEs of 6-dimensional supergravity with 8 real supercharges coupled to any number of vector, tensor and scalar multiplets in all cases. For this we have used the spinorial geometry technique of [11] and the similarity of the KSEs of 6-dimensional supergravity with those of heterotic supergravity. The solutions are uniquely characterized by the isotropy group of the Killing spinors in $Spin(5,1) \cdot Sp(1)$ as given in table 1. This is apart from one case where there is an independent descendant with 3 Killing spinors and isotropy group $Sp(1) \ltimes \mathbb{H}$, table 2.

The geometry of the solutions depends on whether the isotropy group of the Killing spinors is compact or non-compact. In the non-compact case, the spacetime always admits a parallel null 1-form with respect to the connection with skew-symmetric torsion given by the 3-form of the gravitational multiplet. There are backgrounds with $1, 2, 3$ and 4 supersymmetries. The conditions imposed on the fields by the KSEs are given in all cases.

On the other hand if the isotropy group of the Killing spinors is compact, the solutions preserve 2, 4 and 8 supersymmetries. In the case of 2 supersymmetries, the spacetime admits a $3 + 3$ split where the first 3 directions are spanned by 3 parallel vector fields with respect to the connection with skew-symmetric torsion given by the 3-form of the gravitational multiplet. There is also a natural frame on the spacetime given by six 1-form spinor bi-linears. Similarly, the spacetime of solutions with 4 supersymmetries admits a $4 + 2$ split where the 4 directions are spanned by 4 parallel vector fields with respect to a connection with skew-symmetric torsion. The spacetime again admits a natural frame.

In the compact case, the geometry can be further understood provided we take the 3-form field strength of the gravitational multiplet to be closed or assume that the algebra of the vectors fields constructed from spinor bi-linears closes. In such a case, the spacetime can be thought of as principal bundle. For solutions preserving 2 supersymmetries, the spacetime is locally a product $G \times B$, where $G = \mathbb{R}^{3,1}$ or $SL(2,\mathbb{R})$, and B is a 3-dimensional manifold. For solutions preserving 4 supersymmetries, the fibre group has Lie algebra $\mathbb{R}^{3,1}$, $\mathfrak{sl}(2,\mathbb{R}) \oplus \mathfrak{u}(1)$, $\mathbb{R} \oplus \mathfrak{su}(2)$ or $\mathfrak{cw}_4$. Moreover unless the fibre group is abelian, the principal bundle is always twisted over a 2-dimensional base space.

The geometry of 6-dimensional supersymmetric backgrounds is much simpler than those of heterotic supergravity. The most striking simplification occurs in the analysis of the descendants. There is just one independent descendant in 6 dimensions as compared to many possibilities that appear in the heterotic case [13, 14]. It is therefore likely that all half supersymmetric solutions and supersymmetric near horizon geometries of 6-dimensional supergravity can be classified as similar results have been obtained for the heterotic supergravity in [21, 22], see also [4]. However the presence of scalar and vector multiplets in 6 dimensions makes the investigation more involved. Usually such proofs require some delicate additional information about the couplings of these multiplets. Nevertheless, it is likely that such analysis can be carried out under some mild assumptions.

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